Discipline: **Physics**
Subject: **Electromagnetic Theory**
Unit 21:
Lesson/ Module: **Motion in Constant and Uniform Fields**
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Contents

Learning Objectives:

From this module students may get to know about the following:

1.The relativistic motion of a charged particle in a static and uniform electric field.

2.The relativistic motion of a charged particle in a static and uniform magnetic field and expressions for frequency and radius of gyration.

3. Motion in combined, uniform and static, transverse electric and magnetic fields and its use as a velocity selector.

4.*The special case of electric and magnetic fields which are "equal" in magnitude.*

21. Motion in electric and magnetic fields

21.1 Motion in uniform and static electric field

One important field of investigation in electrodynamics is the study of motion of charged particles under the influence of various electric and magnetic field configurations. Such studies are of immense importance in the design of particle accelerators, magneto-hydrodynamics, ionosphere and cosmic rays, to name a few.

We will look at the simplest configurations to begin with, viz., motion in a uniform and static electric field. However, from the very beginning we will consider relativistic dynamics of the charged particle.

Let us say we have a particle of charge q and mass m, passing through a region of uniform and static electric field $\vec{E}$. The force acting on such a particle is $q\vec{E}$. From Newton's second law, the motion of the particle is governed by the equation

$$\frac{d\vec{p}}{dt} = q\vec{E}. \tag{1}$$

Now for a relativistic particle of (rest) mass m, the momentum $\vec{p}$ is given by

$$\vec{p} = \frac{m\vec{v}}{\sqrt{1-\frac{v^2}{c^2}}} = \gamma m\vec{v}, \; \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}. \tag{2}$$

Here, as usual, $\vec{p}$ is the momentum and $\vec{v}$ is the velocity of the particle, and c of course is the velocity of light.

Without any loss of generality we can fix the direction of the electric field to be along one of the axes, say the x-axis

$$\vec{E} = E_x\hat{x}. \tag{3}$$

Let the initial momentum of the particle, $\vec{p}_0$, be in the y direction:

$$\vec{p} = p_0\hat{y}. \tag{4}$$

Our aim is to determine the trajectory of the particle passing through such a region. It is clear that the trajectory will lie in the plane containing the electric field and the initial momentum, i.e., in the x-y plane. Taking the x and y components of (1) and using (3) and (4), we have

$$\frac{dp_x}{dt} = qE_x$$

$$\frac{dp_y}{dt} = 0$$

These equations can be solved immediately to give

$$p_x = qE_x t \ ,$$

$$p_y = p_o \, .$$

Here p_0 is a constant and we have chosen the origin of time so that $p_x = 0$ at $t = 0$. The relativistic expression for the energy of the particle, including the rest energy, is

$$T = \sqrt{m^2c^4 + p^2c^2} \quad = \sqrt{m^2c^4 + p_o{}^2c^2 + (qE_x t)^2 c^2} \quad = \sqrt{T_o{}^2 + (qE_x tc)^2} \ , \qquad (5)$$

T_0 being the initial energy including rest energy.

The relativistic relation between the kinetic energy, momentum and velocity can be written in the form:

$$\vec{v} = \frac{\vec{p}c^2}{T} \ [\ \vec{p} = \gamma m\vec{v}, T = \gamma mc^2\ ]$$

Thus

$$\vec{v} = \frac{d\vec{x}}{dt} = \frac{\vec{p}c^2}{T} \qquad (6)$$

On equating the x and y components,

$$\frac{dx}{dt} = \frac{qE_x c^2 t}{\sqrt{T_o{}^2 + (qE_x ct)^2}} \ ; \ \ \frac{dy}{dt} = \frac{p_o c^2}{\sqrt{T_o{}^2 + (qE_x ct)^2}} \qquad (7)$$

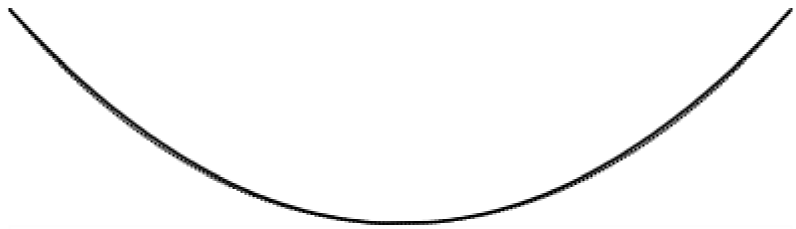
These equations can be solved readily to obtain

$$x(t) = \frac{\sqrt{T_o{}^2 + (qE_x ct)^2}}{qE_x} \ ; \ y(t) = \frac{p_o c}{qE_x} \sinh^{-1}(\frac{qE_x ct}{T_o}) \qquad (8)$$

This is the equation of the trajectory of the particle in a parametric form in terms of time t. On eliminating t from the two equations, we obtain

$$x = \frac{T_o}{qE_x} \cosh(\frac{qE_x y}{p_o c}) \qquad (9)$$

In mathematics, this is the equation of a catenary. This is the shape taken by a uniform chain in gravity, whose two ends are fixed at the same height.



There are certain interesting facts to be noted.

- The force is acting only along the *x*-direction, so the *y* component of the momentum is conserved.

- In the non-relativistic case it implies that the *y* component of velocity is also conserved.

- In the relativistic case, the mass of the particle increases and hence the *y* component of velocity decreases and eventually tends to zero.

- The *x* component of velocity tends to a constant, *c*, and the force essentially increases the mass of the particle.

- In the non-relativistic case the trajectory is well known to be a parabola, and indeed a catenary reduces to a parabola for small *y*.

21.2 Motion in uniform and static magnetic field

The force acting on a particle of mass *m* and velocity $\vec{v}$ in a magnetic field $\vec{B}$ is given by

$$\vec{F} = \frac{d\vec{p}}{dt} = q\vec{v} \times \vec{B} \qquad (10)$$

Since the force is always directed normal to the velocity, *no work is done by the magnetic field* and hence the kinetic energy of the particle, T ($=\gamma mc^2$), also remains unchanged:

$$\frac{dT}{dt} = 0. \qquad (11)$$

Since the kinetic energy is constant in time, so is the magnitude of velocity, and hence so is γ. This leads to considerable simplification of the analysis. Using (2) and the fact that γ is constant, we have

$$\frac{d\vec{p}}{dt} = \frac{d(m\gamma\vec{v})}{dt} = m\gamma\frac{d\vec{v}}{dt} = q\vec{v}\times\vec{B}, \qquad (12)$$

or

$$\frac{d\vec{v}}{dt} = \frac{q}{\gamma m}\vec{v}\times\vec{B} \qquad (13)$$

or

$$\frac{d\vec{v}}{dt} = \vec{v}\times\vec{\omega}_B \qquad (14)$$

where

$$\vec{\omega}_B = \frac{q\vec{B}}{\gamma m} = \frac{qc^2\vec{B}}{T} \qquad (15)$$

ω_B is called *gyration frequency* or *precession frequency*..

Taking the magnetic field $\vec{B}$ to be along the z direction, we have

$$\frac{dv_z}{dt} = 0 \Rightarrow v_z = \text{constant}. \qquad (16)$$

The other two equations are

$$\frac{dv_x}{dt} = v_y\omega_B \; ; \qquad \frac{dv_y}{dt} = -v_x\omega_B \qquad (17)$$

Multiplying the second equation with i and adding the two, we have

$$\frac{d(v_x + iv_y)}{dt} = -i(v_x + iv_y)\omega_B \qquad (18)$$

This equation has the solution

$$v_x = a\omega_B\cos(\omega_B t) \qquad (19)$$

$$v_y = -a\omega_B\sin(\omega_B t) \qquad (20)$$

or

$$\vec{v} = v_z\hat{z} + a\omega_B\cos(\omega_B t)\hat{x} - a\omega_B\sin(\omega_B t)\hat{y} \qquad (21)$$

On integrating, we obtain

$$\vec{x}(t) = \vec{x}_o + v_z t\hat{z} + a\sin(\omega_B t)\hat{x} + a\cos(\omega_B t)\hat{y}\,. \tag{22}$$

The main features of the motion are: [**See Figure]**

- In the *x-y* plane the particle moves in a circle of radius *a* called the *radius of gyration*.

- Along the direction of the magnetic induction there is no force and the particle moves with a uniform velocity.

- The path is a *helix* of radius *a* and *pitch angle* $\alpha = \tan^{-1}(\frac{v_z}{\omega_B a})$.

- The magnitude of the gyration radius *a* depends on the magnitude of magnetic field $\vec{B}$ and the transverse momentum $\vec{p}_\perp$ of the particle:

$$|\vec{p}_\perp| = m\gamma\sqrt{{v_x}^2 + {v_y}^2} = m\gamma a\omega_B = m\gamma a\frac{qB}{\gamma m} = qBa \tag{23}$$

- This provides a way to determine the transverse momentum of a charged particle and is of considerable interest in particle physics where all kinds of stratagems are employed to investigate the properties of particles produced in various particle reactions.

- The motion and nature of trajectory is the same in the relativistic as well as the non-relativistic cases. The path is a helix. The only difference is the appearance of the factor γ in various equations, in particular, the gyration frequency

$$\omega_B = \frac{qB}{\gamma m} = \frac{1}{\gamma}\frac{qB}{m} = \frac{1}{\gamma}(\omega_B)_{NR}\,. \tag{24}$$

21.3 Motion in combined, orthogonal electric and magnetic fields

We now consider a charged particle moving in a combination of electric and magnetic fields. The especially interesting and important case is one in which the two fields are transverse to each other. For a particle with velocity $\vec{v}$, we have the Lorentz force equation and the energy change equation

$$\frac{d\vec{p}}{dt} = q[\vec{E} + \vec{v}\times\vec{B}]\,; \qquad \frac{dT}{dt} = q\vec{v}.\vec{E}\,. \tag{25}$$

The second equation follows from the relation between energy and momentum:

$$T^2 = \vec{p}^2 + c^2 \Rightarrow T\frac{dT}{dt} = \vec{p}.\frac{d\vec{p}}{dt} = q\vec{p}.\vec{E} \Rightarrow \frac{dT}{dt} = \frac{q}{T}\vec{p}.\vec{E} \tag{26}$$

Since $\vec{p} = \gamma m\vec{v}$ and $T = \gamma mc^2$, it follows that $\dfrac{dT}{dt} = q\vec{v}.\vec{E}$. However, since the energy of the particle is no longer constant, we cannot obtain a simple equation for the velocity, as was done for the case of magnetic field alone. We now take an altogether different approach and appeal to Lorentz transformations. The electric and magnetic field together form a tensor and transform under Lorentz transformation as the components of a second rank tensor. Let the given frame of reference be K. Consider another frame of reference K' moving with velocity $\vec{u}$ with respect to the frame K. Then in the frame K', the Lorentz force equation, the electric field and the magnetic field are transformed to

$$\frac{d\vec{p}'}{dt'} = q[\vec{E}' + \vec{v}' \times \vec{B}'] \tag{27}$$

$$\vec{E}' = \gamma(\vec{E} + c\vec{\beta} \times \vec{B}) - \frac{\gamma^2}{\gamma+1}(\vec{\beta}.\vec{E})\vec{\beta}; \tag{28}$$

$$\vec{B}' = \gamma(\vec{B} - \vec{\beta} \times \vec{E}/c) - \frac{\gamma^2}{\gamma+1}(\vec{\beta}.\vec{B})\vec{\beta};\ \vec{\beta} = \vec{u}/c. \tag{29}$$

There are three cases to be considered: electric field (divided by *c*; remember *E/c* has the same dimensions as *B*) less than, greater than or equal to the magnetic field.

21.3.1 |E| /c< |B|

In this case choose

$$\vec{u} = \frac{\vec{E} \times \vec{B}}{B^2} \Rightarrow \vec{\beta} = \frac{\vec{E} \times \vec{B}}{cB^2} \Rightarrow \left|\vec{\beta}\right| = E/(cB) \tag{30}$$

Since $\left|E\right|/c < \left|B\right|$, therefore $\left|\vec{\beta}\right| < 1$ and this is a valid Lorentz transformation. Then in K'

$$\vec{E}' = \gamma(\vec{E} + \frac{\vec{E} \times \vec{B}}{B^2} \times \vec{B}) - \frac{\gamma^2}{\gamma+1}(\vec{\beta}.\vec{E})\vec{\beta}$$

$$= \gamma(\vec{E} + \frac{(\vec{E}.\vec{B})\vec{B}}{B^2} - \vec{E}) - \frac{\gamma^2}{\gamma+1}(\vec{\beta}.\vec{E})\vec{\beta} = 0 \tag{31a}$$

since $\vec{E}, \vec{B}, \vec{\beta}$ are all mutually orthogonal. Similarly,

$$\vec{B}' = \gamma(\vec{B} - \frac{\vec{E} \times \vec{B}}{c^2B^2} \times \vec{E}) - \frac{\gamma^2}{\gamma+1}(\vec{\beta}.\vec{B})\vec{\beta} = \gamma[\vec{B} - \frac{E^2}{c^2B^2}\vec{B}]$$

$$= \gamma[1 - \frac{E^2}{c^2B^2}]\vec{B} = \vec{B}/\gamma \tag{31b}$$

Thus in the frame K', the field is only static, uniform magnetic field (the electric field has simply vanished!), in the same direction as in K but reduced by a factor of γ. The trajectory in the K' frame is thus a spiral around the lines of force. In the original frame of reference K, this gyration around the lines of force is accompanied by a uniform drift $\vec{u}$ normal to both the fields. This is often called the $E \times B$ *drift*. The direction of the drift is independent of the sign of the charge of the particle.

This rather strange looking result where the particle drifts in a direction normal to both electric and magnetic field can be understood in a qualitative way. A particle that starts gyrating around $\vec{B}$ is accelerated by $\vec{E}$, gains energy, and moves in a path with larger radius for roughly half of its cycle. Remember, the radius of gyration is proportional to the transverse momentum! In the other half, $\vec{E}$ decelerates it, causing it to lose energy and so move in a tighter arc. The combination of the two arcs produces a translation normal to both $\vec{E}$ and $\vec{B}$. **[See Figure, Figure 12.2 Jackson Edition 2]**

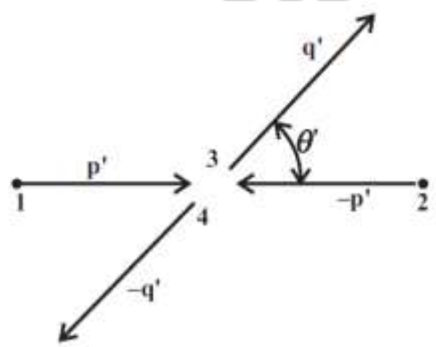


Fig. 12.2 Momentum vectors in the center of momentum frame for elastic scattering or a two-body reaction.

21.3.2 The velocity selector

Let a particle enter a region of crossed electric and magnetic fields with $|E| < c|B|$. If the particle enters this region with a speed $\vec{u} = \dfrac{\vec{E} \times \vec{B}}{B^2}$, normal to both the fields, then in the K' frame moving with velocity $\vec{u}$ with respect to K, the speed of the particle is zero. Since the electric field is also zero in this frame, there is no force acting on the particle in K'; its speed remains zero in K', which implies that its speed will remain $\vec{u}$ in the frame K which further implies that the particle will move out *undeviated* through the crossed $\vec{E}$ and $\vec{B}$ fields. Suitable entrance and exit slits will then allow only a very narrow band of velocities around *E/B* to be transmitted. This is a very useful way of having a velocity selector.

Combined with momentum selectors like a deflecting magnet, the $\vec{E}\times\vec{B}$ velocity selectors can extract a very pure and monoenergetic beam with different masses and momenta – commonly used in high-energy accelerators.

21.3.3 |E| > c|B|

In this case choose

$$\vec{u} = \frac{\vec{E}\times\vec{B}}{E^2} \tag{32}$$

Now $|E|/c > |B|$, therefore $|\vec{\beta}| < 1$ and this is again a valid Lorentz transformation. An analysis similar to the $|E| < c|B|$ case gives in K'

$$\vec{B}' = 0; \qquad \vec{E}' = \vec{E}/\gamma\ . \tag{33}$$

In K' the field is pure uniform and static electric field; the motion is a catenary with ever-increasing velocity. The motion in K can be obtained by using the relativistic velocity transformation law.

21.4 The special case of equal magnitudes: E=cB

In this case it is not possible to make transformation to a frame of reference in which the field is purely electric or purely magnetic and the problem has to be solved in full generality. Let the magnetic field $\vec{B}$ be in the *z*-direction and the electric field $\vec{E}$ be in the y-direction. Let the magnitude of the electric field be denoted by *E*. Then $\vec{E} = E\hat{y}$, $\vec{B} = \frac{1}{c}E\hat{z}$ and

$$\frac{d\vec{p}}{dt} = q[\vec{E} + \vec{v}\times\vec{B}] = q[E\hat{y} + \frac{v_y}{c}E\hat{x} - \frac{v_x}{c}E\hat{y}] \tag{34}$$

or

$$\frac{dp_x}{dt} = \frac{q}{c}Ev_y \tag{35}$$

$$\frac{dp_y}{dt} = qE(1-\frac{v_x}{c}) \tag{36}$$

$$\frac{dp_z}{dt} = 0 \Rightarrow p_z = \text{constant} \tag{37}$$

Further, if *T* is the kinetic energy, from equation (25) and (35)

$$\frac{dT}{dt} = q\vec{v}.\vec{E} = qEv_y = c\frac{dp_x}{dt} \qquad (38)$$

or

$$\frac{d}{dt}(T - cp_x) = 0 \Rightarrow T - cp_x = \alpha \qquad (39)$$

where α is some constant. Now

$$(T^2 - c^2 p_x{}^2) = (T - cp_x)((T + cp_x) = \alpha(T + cp_x) \qquad (40)$$

We have the relativistic relation

$$T^2 = m^2c^4 + c^2 |\vec{p}|^2 = m^2c^4 + c^2 p_z{}^2 + c^2(p_x{}^2 + p_y{}^2) = T'^2 + c^2(p_x{}^2 + p_y{}^2) \qquad (41)$$

where

$$T'^2 = m^2c^4 + c^2 p_z{}^2 \qquad (42)$$

is a constant since p_z is also a constant. On using equations (40) and (41) we have

$$T'^2 + c^2 p_y{}^2 = T^2 - c^2 p_x{}^2 = \alpha(T + cp_x)$$

Thus

$$(T + cp_x) = \frac{1}{\alpha}(T'^2 + c^2 p_y{}^2) \qquad (43)$$

On solving equations (39) and (43) for *T* and p_x, we get

$$T = \frac{c^2 p_y{}^2 + T'_2}{2\alpha} + \frac{\alpha}{2} \qquad (44)$$

$$p_x = \frac{c^2 p_y{}^2 + T'_2}{2\alpha c} - \frac{\alpha}{2c} \qquad (45)$$

On multiplying equation (36) with *T* and using equation (39) and the relation $Tv_x = \gamma mc^2 v_x = c^2 p_x$, we have

$$T\frac{dp_y}{dt} = qE(T - \frac{Tv_x}{c}) = qE(T - cp_x) = qE\alpha \qquad (46)$$

or

$$2qE = \frac{2}{\alpha} T \frac{dp_y}{dt} = \frac{1}{\alpha}[(T + cp_x) + (T - cp_x)]\frac{dp_y}{dt} = \frac{1}{\alpha}[\alpha + (T + cp_x)]\frac{dp_y}{dt}$$

$$= [1 + \frac{(T + cp_x)}{\alpha}]\frac{dp_y}{dt} = [1 + \frac{(T'^2 + c^2 {p_y}^2)}{\alpha^2}]\frac{dp_y}{dt} = [1 + \frac{T'^2}{\alpha^2}]\frac{dp_y}{dt} + \frac{c^2 {p_y}^2}{\alpha^2}\frac{dp_y}{dt}$$

On integrating the last form of the above relation, we have

$$2qEt = (1 + \frac{T'^2}{\alpha^2}) p_y + \frac{c^2}{3\alpha^2} {p_y}^3 \tag{47}$$

To determine the trajectory, we start with

$$\frac{dx}{dt} = v_x = \frac{c^2 p_x}{T} \Rightarrow dx = \frac{c^2 p_x}{T} dt = \frac{c^2 p_x}{T}\frac{1}{dp_y / dt} dp_y = \frac{c^2}{qE\alpha} p_x dp_y$$

$$= \frac{c^2}{qE\alpha}(-\frac{\alpha}{2c} + \frac{c^2 {p_y}^2 + T'^2}{2\alpha c}) dp_y$$

On integrating this equation we get

$$x = \frac{c^2}{qE\alpha}(-\frac{\alpha}{2c} + \frac{T'^2}{2\alpha c}) p_y + \frac{c^2}{qE\alpha}\frac{c}{2\alpha}\frac{1}{3}{p_y}^3$$

Or

$$x = \frac{c}{2qE}(-1 + \frac{T'^2}{\alpha^2}) p_y + \frac{c^3}{6qE\alpha^2} {p_y}^3 \tag{48}$$

$$\frac{dy}{dt} = v_y = \frac{c^2 p_y}{T} \Rightarrow dy = \frac{c^2 p_y}{T} dt = \frac{c^2 p_y}{T}\frac{1}{dp_y / dt} dp_y = \frac{c^2}{qE\alpha} p_y dp_y$$

On integrating

$$y = \frac{c^2}{2qE\alpha} {p_y}^2 \tag{49}$$

Similarly

$$dz = \frac{c^2}{qE\alpha} p_z dp_y$$

Therefore

$$z = \frac{c^2 p_z}{qE\alpha} p_y \qquad (50)$$

Formulas (48) – (50) determine the motion of the particle in parametric form in which p_y acts as the parameter. The time dependence is given by equation (47).

Thus the problem can in principle be fully solved. The solution looks very unlike what we get for the other two cases that we discussed above, viz., $E/c<B$ and $E/c>B$.

Summary

1. *The relativistic motion of a charged particle in a uniform and constant electric field was studied. The differences from the non-relativistic case were discussed.*

2. *The relativistic motion of a charged particle in a static and uniform magnetic field was considered. Expressions for the radius and the frequency of gyration were obtained.*

3. *Next motion in combined, uniform and static, transverse electric and magnetic fields was considered.*

4. *For the case of $|E|<c|B|$, a transformation could be made to another frame of reference in which the field is purely electric in nature and the solution is thus a catenary in that frame. The motion in the original frame is obtained by applying the law of addition of velocities.*

5. *In the case of $|E|<c|B|$, a transformation can be made to a frame of reference in which the field is purely magnetic and the solution is thus similar to that in a pure magnetic field along with a "drift" in the transverse direction. The use of such a configuration as a velocity selector was described.*

6. *Finally the special case of electric and magnetic fields which are equal in magnitude, i.e.,* $|E|=c|B|$ *was discussed and the complete solution obtained.*